where the bar force increment ${}_1F$ is

$$ {}^2_1F = f'\left({}^1_0\epsilon\right){}^1A\,({}^2L^2 - {}^1L^2)/(2\,{}^1L^2) \tag{18} $$

The equations derived in this section for the bar forces [Eqs. (11), (14), (17),] will be referred to as the "incremental form," which are piecewise linear, whereas those derived previously based on the integration process [Eqs. (4), (2), (6), (8)] will be termed the "total form," which are fully nonlinear.

Numerical Examples

The two-member truss shown in Fig. 1 will be studied, assuming ${}^0A = {}^1A = {}^2A = 10$ cm^2, ${}^0L = 50$ cm, and ${}^mL = 25\,(1 + \cos\alpha)$ cm. Two types of materials will be considered: 1) Case A—linearly elastic materials (in TL sense):

$$ f'\left({}^i_0\epsilon\right) = E \tag{19} $$

and 2) Case B – nonlinearly elastic materials:

$$ f'\left({}^i_0\epsilon\right) = E + 2E_a\left|{}^i_0\epsilon\right| \tag{20} $$

where E and E_a are the material constants. For the truss with $\alpha = 30$ deg and $E = 1$ MPa, the tangent moduli implied by the TL, UL, and GL formulations, i.e., Eqs. (19), (5), and (7), are drawn in Fig. 1. As can be seen, the material implied by the UL formulation is in fact nonlinear. It tends to soften in tension and to harden in compression. Although the materials implied by the TL and GL formulations are linear, they are not identically the same.

In the incremental load-deflection analysis, the truss is assumed to have an angle of $\alpha = 60$ deg with $E_a = 10E = 10$ MPa. The method of solution adopted herein is the displacement control method. Both the "total form" and "incremental form" will be used for calculating the member forces. The results obtained with the "total form" will be referred to as exact, as they are not affected by the step sizes used. Figures 2 and 3 show the solutions obtained with the UL formulation for case A materials and those by the TL formulation for case B materials, respectively. Additional examples including more complicated trusses may be found in Ref. 4.

With regard to the previous solutions, the following observations can be made: 1) For formulation with "nonlinear" equivalent tangent modulus in the TL sense, the load-deflection curves obtained with the "incremental form" are step-size dependent (Figs. 2 and 3), because of the linearization of the material laws in each step; 2) For formulations with "linear" equivalent tangent modulus in the TL sense, the solutions obtained with the "incremental form" coincide with the exact ones, which are step-size independent[4]; and 3) All of the solutions that are step-size dependent converge to the exact ones as the step sizes are reduced.

Conclusions

In postbuckling analysis of trusses with large strains, the use of identical constants in the material laws [Eqs. (1)] does not imply identical tangent modulus for the TL, UL, and GL formulations. Two procedures have been presented for calculating the bar forces. One is the "total form," which is fully nonlinear, and the other is the "incremental form," which is a piecewise linear approximation of the former. Using the "total form" in a nonlinear analysis will result in solutions that are exact and step-size independent. In contrast, using the "incremental form" for bar forces will generate solutions that are step-size dependent. For problems with large strains, the errors involved with the latter procedure can be significant.

Acknowledgment

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Two-Level Approximation Method for Stress Constraints in Structural Optimization

Garret N. Vanderplaats*
and
Srinivas Kodiyalam†
VMA Engineering
Goleta, California

Introduction

RECENT trends in stress-constrained optimization have established that approximation of the element nodal forces better retains the essential nonlinearities of the true (original) constraints than the usual approach of linearizing the actual constraints with respect to the design variables.[1,2] However, for continuum structures, this increases the cost of the approximate optimization phase because numerous element level solutions of the equilibrium equations are required for stress recovery.

Here both the force approximation and the direct-stress approximation methods are considered together to create a two-level scheme for stress constraints. This is an extension of the work reported in Ref. 3.

Mathematical Problem Statement

The mathematical programming problem is stated as follows.

Minimize

$$ F(X) \qquad \text{weight of the structure} \tag{1} $$

Subject to

$$ g_j\,(X) \le 0 \qquad j = 1, M \qquad \text{stress constraints} \tag{2} $$

$$ \mathbf{X}_i^L \le \mathbf{X}_i \le \mathbf{X}_i^U \qquad \mathbf{i} = 1, N \qquad \text{side constraints} \tag{3} $$

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*President. Associate Fellow AIAA.
†Research Engineer. Member AIAA.

where X is the vector of design variables. The stress constraints considered here are of the form of octahedral shear stresses at various locations of the finite element.

Two-Level Approximation Method

The first level of approximations is the force approximation procedure detailed in Ref. 2, where the element nodal forces are expanded in Taylor series and the linearized forces thus obtained are in turn used to compute the approximate stresses. An elemental approach is used to avoid repeated solutions of a large system of equations to recover the stresses from the approximate forces. For a 20-node solid element, this elemental approach requires k number of solutions of 54×54 equations where k represents the total number of elements in the finite-element model. For structures modeled with a large number of elements, this step is time consuming. For an approximate iteration, it is sufficient if we consider only those elements which model the portion of the structure that are critically stressed, thus reducing the computational effort. This "critically stressed" region can be simply identified by those individual elements that include constraints in the set J_R.

The set of retained constraints J_R is given by

$$J_R = \{(g_k, k \epsilon Q_1), (g_i \geq -0.25, i \epsilon Q_1)\} \tag{4}$$

where Q_1 = the first p (p is user defined) critical constraints and has a maximum of three constraints for any single finite element. For those elements that contain a retained constraint, the element joint forces are approximated for the loading condition corresponding to this constraint. This defines the first level of the approximate problem:

$$F(X) \qquad \text{retained in its original form} \tag{5}$$

$$\tilde{F}_{ij} = F_{ij}\,(X^0) + \nabla F_{ij}\,(X^0) \cdot \delta X \tag{6}$$

where i is the element number, j the loading condition, and $\delta X = X - X^0$. Here, $F(X)$ is the weight of the structure, and $\tilde{F}_{ij}$ are the approximations to the element forces.

Whenever stresses must be calculated, based on this approximation, it is first necessary to delete the rigid-body degrees of freedom from the element and then solve for the element level displacements using

$$k^e u^e = \tilde{F}^e \tag{7}$$

The stresses are then recovered from these local displacements. If this process is carried out within the approximate optimization process, this can require considerable computational effort. The approach used here is to create a second-level approximation based on these approximate forces and to use this for optimization.

An optimization strategy is now used to create and solve the optimization problem based on this approximate analysis, thus creating the second level of approximations. In this study, the Sequential Linear Programming (SLP) Method was used to create the second-level problem, which can be mathematically stated as

Minimize

$$\tilde{F}(X) = F(x^0) + \nabla F(X^0) \cdot \delta X \tag{8}$$

Subject to

$$\tilde{g}_j\,(X) = \tilde{g}_j\,(X^0) + \nabla \tilde{g}_j\,(X^0) \cdot \delta X \leq 0 \qquad j \epsilon J_R \tag{9}$$

$$\delta X_i^L \leq \delta X_i \leq \delta X_i^U \qquad i = 1, N \tag{10}$$

where $F(X^0)$ and $\tilde{g}_j(X^0)$ represent the objective and stress constraint function values at X^0 (the initial design), respectively. On the first iteration of the second-level approximate optimization, these are precise. On subsequent iterations, the $\tilde{g}_j(X^0)$ are based on the $\tilde{F}_{ij}$ (of the first-level approximation). During each iteration of the second-level SLP problem, the $\tilde{F}_{ij}$ are held invariant at values computed using the first-level approximation.

The key point to note here is that the constraints, given in Eq. (9), are updated based on the nodal-force approximations rather than a complete finite-element analysis. The gradients of the stresses used here to obtain the linear approximation to the constraints are from the stresses which were, in turn, obtained from the force approximations.

The problem given in Eqs. (8–10) is linear in the design variables X and can be solved for X using linear programming or other efficient optimization algorithms. Here, the Modified Method of Feasible Directions, programmed in the design optimization tools (DOT) optimization code,[4] is used to solve the approximate problem. The design is updated, and a new linearization is performed repeating the process until convergence is achieved. The move limits of Eq. (10) are imposed to limit the design to the region of applicability of the approximation and also to prevent an unbounded solution.

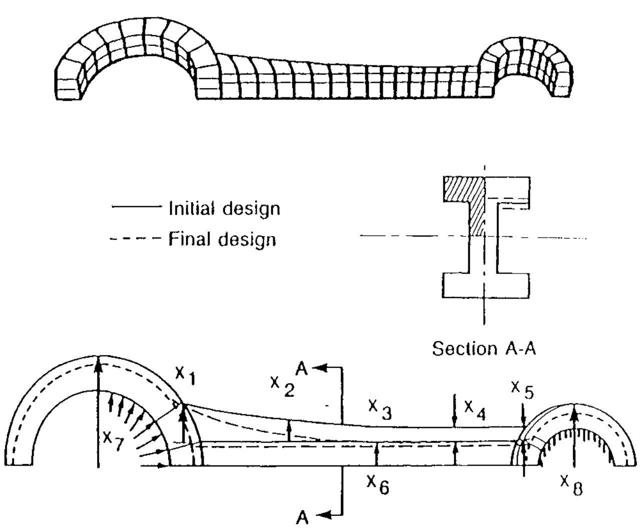
The difference between the one-level (force approximation) and the two-level problems can be outlined as in Table 1.

Table 1 Comparison of One- and Two-Level Approximations

One level	Two level
Linearize element nodal forces: $\tilde{F}^e = F^e(X^0) + \nabla F^e(X^0) \cdot \delta X$	Linearize element nodal forces: $\tilde{F}^e = F^e(X^0) + \nabla F^e(X^0) \cdot \delta X$
Solve approximate problem: when stresses are needed, update $\tilde{F}^e$ and compute stresses and their gradients from $\tilde{\sigma}^e = f(\tilde{F}^e)$	Compute approximate stresses and their gradients: $\tilde{\sigma}^e = f(\tilde{F}^e)$ and $\nabla \tilde{\sigma}^e = (f\tilde{F}^e)$
	Create the linear approximations of objective $F(x)$ and constraint functions
Check for convergence; if satisfied, terminate, otherwise, iterate	Solve approximate problem: when stresses are needed, calculate from preceding approximation
	Check for convergence; if satisfied terminate, otherwise, iterate.

Numerical Example

The two-level approximation method just outlined is implemented on a Cray-XMP/48 through a research computer program. An engine connecting rod is solved for optimum shape using the proposed method.



Note: X's are design variables

Fig. 1 Design model of engine connecting rod.

Table 2 Initial and final designs for the engine connecting rod

Design Variable	Initial	Final: Two-level method	Final: One-level method
1	10.9560	13.3594	12.4535
2	6.3700	3.3057	2.3443
3	3.9667	1.6518	1.5074
4	3.0024	1.1620	0.7928
5	3.2711	1.2755	1.4399
6	6.8156	5.2980	5.4361
7	31.271	25.8998	25.9972
8	17.553	13.3000	13.3000
Objective, mm^3	15562.4	6760.6	6528.7
Iterations	—	4	6
Maximum constraint value	−0.51	0.0015	0.0010

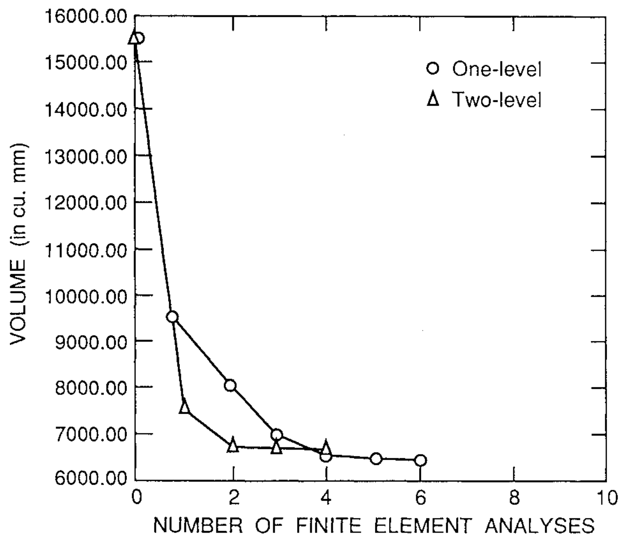


Fig. 2 Volume design history.

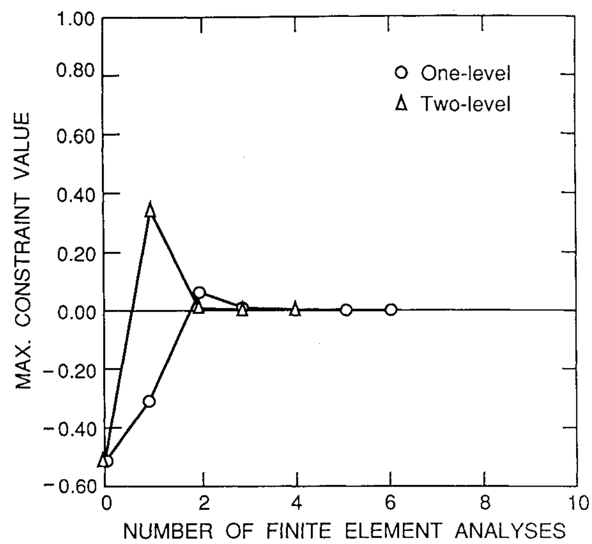


Fig. 3 Constraint design history.

Table 3 CPU time

Example	Two-level method	One-level method
	Total CPU time	
Connecting rod	12 min 32 s	39 min 04 s
	CPU time spent in approximate optimization phase	
Connecting rod	5 min 02 s	23 min 22 s

A move limit of 80% of the initial design was used at the start of the optimization process. At the end of each approximate problem solution, if any of the constraints are more violated than the previous iteration, the move limits on each of the design variables were reduced by 50%. Move limits were never reduced to less than 25%. The use of large move limits is possible because the force approximation method (first-level approximations) has a wide applicable range in the design variable space.

The connecting rod, shown in Fig. 1 (a one-fourth model), is modeled with 105 solid, 20-node isoparametric elements and has a total of 2126 degrees of freedom. The design and analysis model and the loading conditions are the same as in Ref. 3.

The optimization results are given in Table 2, and Figs. 2 and 3 show the volume and stress constraint history.
The CPU time taken on a Cray-XMP/48 to solve this example is given in Table 3. Both the gradients of the nodal forces and the stresses were calculated using the forward-finite difference method.

Conclusions

A two-level approximation method has been outlined where at the second stage the stress constraints are directly expanded in Taylor series with respect to the shape variations. This approximation is nested within the original force approximation. These nested approximations along with constraint deletion techniques have reduced the computational effort significantly. The computational efficiency improvements are mainly attributed to bypassing the solutions for the element-level displacements at the second-stage of approximations.

The present method converges at a slightly higher objective function compared to the one-level approximation approach. This difference is considered to be within the convergence tolerance of the method.

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Optimal Design for Plate Structures with Buckling Constraints

Chien-Chang Lin and Ine-Wei Liu†
National Chung-Hsing University, Taichung, Taiwan

Introduction

THE first use of the optimality criterion as a redesign equation was due to Venkayya et al.,[1] and represented a significant development in structural optimization. However, the assumption of linear size-stiffness relations in their algorithm was a major restriction. Discretized optimality criteria methods that treat system buckling and minimum size constraints have been reported,[2,3] but these studies fail to treat system buckling in parallel with other common behavioral constraints such as static stress and displacement limitations. Furthermore, they completely ignore the more important problems associated with preventing local buckling failure modes. On the other hand, Giles[4] includes consideration of buckling failure modes in the design of wing structures. This Note presents a general redesign approach that includes the usual stress, displacement, member size, and buckling constraints as discussing in Ref. 5. The optimality criterion derived for all the constraints imposed on the structure is equivalent to the Kuhn-Tucker conditions of nonlinear mathematical programming. However, the criterion approach is formulated in a general form with all the constraints being combined together or in a simple form with single constraint. An optimization computer code ARS 4 (automatic resizing system 4) has been developed and applied to some practical examples.

Analysis

The objective of our problem is to find the size vector t of ne elements t_i $(i=1,2,\ldots,ne)$, which will minimize the plate structures weight function $W(t)$ subject to

Behavior constraints:

$$g_j(t) = G_j(t) - \hat{G}_j \leq 0, \quad j=1,\ldots,nc \tag{1a}$$

Size constraints:

$$t_i^L \leq t_i \leq t_i^U, \quad i=1,\ldots,ne \tag{1b}$$

The behavior quantities $G_j(t)$ are displacements, stresses, and critical buckling load parameter, whereas $\hat{G}_j$ are limits. It is seldom desirable to have each t_i as an independent design variable. Design variable linking can be used to reduce the number of design variables to nv, $nv \leq ne$; therefore, the actual design variables, denoted by $x = \{x_1, x_2,\ldots, x_{nv}\}^T$, are related to t_i by a relationship as

$$t_i = \zeta_{im} x_m \qquad (i=1,\ldots,ne; \quad m=1,\ldots,nv) \tag{2}$$

It is well known to define the linear instability of a structure by the eigenvalue problem, i.e.,

$$(K - \mu_j K_G)\bar{\eta}_j = 0 \tag{3}$$

where K is the structural stiffness matrix, K_G the geometric stiffness matrix, and $\bar{\eta}_j$ the eigenvector associated with the jth eigenvalue μ_j. The eigenvalue μ_j can be obtained by the Rayleigh quotient as

$$\mu_j = \frac{\bar{\eta}_j^T K \bar{\eta}_j}{\bar{\eta}_j^T K_G \bar{\eta}_j}, \qquad j=1,\ldots,nd \tag{4}$$

To solve the optimization problem, we first select the initial point in the feasible region and then, after new iterations, move the design point toward the constrained boundary. Assume there are na constraints encountered during the optimization process, i.e.,

$$g_j(x) = 0, \quad j=1,\ldots,na \qquad \text{(active constraints)} \tag{5}$$

$$g_j(x) \leq 0, \quad j=na+1,\ldots,nc \qquad \text{(passive constraints)} \tag{6}$$

By the Kuhn-Tucker optimality criterion, at the optimum point x^*, we may find a set of nonnegative Lagrange multipliers $\bar{\lambda} = \{\lambda_1,\lambda_2,\ldots,\lambda_{na}\}^T$, such that

$$-\frac{\partial W(x^*)}{\partial x_i} = \sum_{j=1}^{na} \lambda_j \frac{\partial g_j(x^*)}{\partial x_i}, \quad i=1,\ldots,nv \tag{7}$$

The gradient projection method can be used to solve Eq. (7) instead of the approximate approach. If the optimum point has been found, then Eq. (7) must be achieved. In general, during the designing iterations, the ratio of the right side to the left side of each part in Eq. (7) might not be 1 but will approach 1. We define multiplier update factors r_i in the process of optimization by

$$r_i = -\sum_{j=1}^{na} \lambda_j \frac{\partial_{gi}/\partial x_i}{\partial W/\partial x_i}, \qquad i=1,\ldots,nv \tag{8}$$

and use them as modification factors for the iterations such that

$$x_i^{k+1} = x_i^k r_i, \qquad i=1,\ldots,nv \tag{9}$$

assuming that all the design variables are active and k is the iteration number. As the values of r_i converge to one, the optimum point is reached.

Design sensitivity analysis plays a central role in structual optimization, since virtually all optimization methods require the computation of derivatives of structural response quantities with respect to design variables. In optimality criterion methods, these derivatives are required in calculating the Lagrange multipliers. The detailed sensitivity analysis for the displacements, stresses, and buckling constraints used in this study can be found in Ref. 5.

Design Algorithm

The most important design requirement in a structure is to satisfy the appropriate strength criterion in each element. In practice, the strength criterion is satisfied by using the fully stressed design (FSD) concept, which is one of the early optimality criteria. When the design point is far from the optimal point, the use of FSD can made the design variables rapidly approach the vicinity of the optimal point. As the designing

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*Professor, Institute of Applied Mathematics. Member AIAA.
†Graduate Student, Institute of Applied Mathematics.